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SPLINE APPROXIMATION TO THE SOLUTION
OF A CLASS OF ABEL INTEGRAL EQUATIONS.

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ABSTRACT

Global approximation methods for solving the Abel integral equation

$$\int_0^x \frac{1}{(x-s)^\alpha} y(s)\,ds = f(x) , \qquad 0 < \alpha < 1 , \quad x \geq 0 ,$$

by means of splines with full continuity are considered. The methods are based on using the differentiated form of the above equation. It is shown that the use of linear splines in C leads to a $2-\alpha$ method for $0 < \alpha < 1$ and the use of quadratic splines in C^1 leads to a $3-\alpha$ method, which computational experiments indicate is stable for $0 < \alpha < 1$, though this is proved only for $0.415 \ (\doteq 2 - \frac{\ln 3}{\ln 2}) \leq \alpha < 1$. The same technique applied to cubic and higher-order splines gives rise to divergent methods.

AMS(MOS) Subject Classification: 45E10, 45L10, 65R05

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           Quadratic splines, Cubic splines, Asymptotic error estimates,
           Higher order method.

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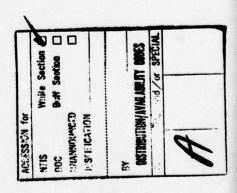
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Significance and Explanation

Abel-type integral equations (see Abstract) occur very often in applications. Typical examples are the solution of the falling particle problem, measurement in axially symmetric systems, and the analysis of Brownian motion and diffusion processes such as heat-conduction.

In this paper, a higher order global approximation method has been developed for the solution of a class of Abel integral equations (see Abstract), using quadratic splines with continuous first derivative. The method is based on using the differentiated form of the given equation alone. It is self-starting and the step-size can be changed at any knot of the spline without added complication. At each step only one equation has to be solved, as compared with other higher order methods which require solution of a system of equations whose coefficients usually involve a much larger number of integrals to be evaluated. The computational effort required by our method is only marginally greater than that required for a linear spline solution of the original equation. Convergence is obtained not only for the approximate solution but also for its first two derivatives. Thus the method is economical when the values of the solution and its derivatives are required at a large number of points where usual discrete methods of computation will be time consuming. We have also derived very simple asymptotic error formulas for the first and second derivatives of our approximate solution.

SPLINE APPROXIMATION TO THE SOLUTION

OF A CLASS OF ABEL INTEGRAL EQUATIONS

Hing-Sum Hung

1. <u>Introduction</u>. In this paper we consider the Abel integral equation

(1.1)
$$\int_0^x \frac{1}{(x-s)^\alpha} y(s)ds = f(x) \ , \quad 0 < \alpha < 1 \ ,$$

which is to be solved for $y(x)$ in $0 \leq x \leq 1$. This is a classical equation, and it arises

in many mathematical and physical problems (see, for example, Noble [8] and Anderssen [1]).

If it is assumed that $f(x)$ is differentiable, then the solution of (1.1) is explicitly

given by [9]

(1.2)
$$y(x) = \frac{\sin \alpha\pi}{\alpha\pi} \left[\frac{d}{dx} \int_0^x \frac{f(s)}{(x-s)^{1-\alpha}} ds \right]$$

$$= \frac{\sin \alpha\pi}{\pi} \left[\frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(s)}{(x-s)^{1-\alpha}} ds \right] \ .$$

The literature on the numerical analysis of Equation (1.1) and related equations is not

only extensive but also contains a great variety of mathematically independent proposals

for their solution numerically. A bibliography can be found in [1].

Recently, Equation (1.1) has been considered by Weiss and Anderssen [11] and by Weiss

[10], who applied product integration techniques to generate approximate values for $y(x)$

at a set of discrete points by means of piecewise constant functions and linear splines.

Benson [2] uses the same methods to a more general equation, proves convergence and obtains

asymptotic error estimates. Their approach, however, does not allow the order of the method

to go beyond two theoretically. Brunner [3] examines a direct method for solving Equation

(1.1) which makes use of spline functions where the usual continuity requirements are some-

what relaxed. To be precise, he uses piecewise polynomials of a given degree and of class

C to generate approximate solutions for Equation (1.1), and shows that convergence of order

m is obtained if m^{th} degree piecewise polynomials are used. His method is basically a

block by block method in the sense of [7] which requires at each step the solution of a

system of equations.

The aim of the present paper is to use our concepts developed in [5] to obtain a

global approximate solution for Equation (1.1) by quadratic splines in C^1. The method

to be described in Section 2 is based on using the differentiated form of (1.1) alone. It is a self-starting step by step method; the stepsize can be changed at any knot of the spline without added complication. At each step, only one equation has to be solved. The computational effort required is only marginally greater than that required for a linear spline solution of the original Equation (1.1) (see, for instance, [2] or [10]). In Section 3, we first derive asymptotic error estimates for the first and second derivatives of the approximate solution, then we prove that the orders of convergence for the approximate solution as well as its first two derivatives at each point in the interval of integration are $3-\alpha$, 2 and 1, respectively, for $0.415 (\doteq 2 - \frac{\ln 3}{\ln 2}) \le \alpha < 1$. This interval contains the important case $\alpha = \frac{1}{2}$. Numerical examples are given in Section 4. Finally, we show by an example, in Section 5, that the same technique applied to cubic splines in C^2 leads to a divergent method. A $2-\alpha$ method for solving (1.1) by linear splines is also considered. The results are summarized in the appendix of this paper.

In the present paper it is assumed that $0 \le x \le 1$, but this restriction is not essential.

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2. **The Quadratic Spline Method.** Let $x_i = ih$, $i = 0,1,2,\ldots$, where h is an arbitrary constant stepsize. Let $y(x)$ be the exact solution of (1.1), and let Y_i, Y_i' denote approximations to $y(x_i)$, $y'(x_i)$, respectively. We use a quadratic spline $P(x)$ in $C^1[0,1]$ with knots at the points x_i as an approximation to $y(x)$, i.e., for $i = 0,1,2,\ldots$

$$(2.1) \qquad P(x) = Y_i + \frac{h}{2} [(2u_i(x) - u_i^2(x))Y_i' + u_i^2(x)Y_{i+1}'], \quad x_i \le x \le x_{i+1} \ ,$$

where $u_i(x) = (x-x_i)/h$. Differentiating (2.1) we obtain the derivative of $P(x)$,

$$(2.2) \qquad P'(x) = u_i(x)Y_{i+1}' - u_{i+1}(x)Y_i', \quad x_i \le x \le x_{i+1} \ .$$

Both $P(x)$ and $P'(x)$ are continuous at the knots.

In order to obtain a numerical solution of (1.1) by using the quadratic spline (2.1), we have to introduce the differentiated form of (1.1). Assume that we can differentiate (1.1). Then

$$(2.3) \qquad \int_0^x \frac{1}{(x-s)^\alpha} y'(s)ds = f'(x) - y(0)x^{-\alpha}.$$

The approximate solution $P(x)$ of the integral equation (1.1) is derived from values $P(x_i) = Y_i$ and $P'(x_i) = Y_i'$. We know how to find Y_{k+1}', assuming that we know Y_i', $i = 0,1,\ldots,k$. We can then deduce Y_{k+1} from the equation obtained by setting $x = x_{k+1}$ in (2.1), which gives

$$(2.4) \qquad Y_{k+1} = Y_k + \frac{h}{2} (Y_k' + Y_{k+1}') \ .$$

To compute Y_i' , we require that $P'(x)$ satisfies (2.3), i.e., $y'(x)$ is replaced by $P'(x)$ derived from the values $P'(x_i) = Y_i'$ computed from

$$(2.5) \qquad \int_0^{x_k} \frac{1}{(x_k-s)^\alpha} P'(s)ds = f'(x_k) - y(0)x_k^{-\alpha} , \quad k = 1,2,\ldots \ .$$

This can be rewritten in the form:

$$(2.6) \qquad \sum_{i=0}^k w_{k,i}Y_i' = f'(x_k) - y(0)x_k^{-\alpha} , \quad k = 1,2,\ldots \ ,$$

where

$$\begin{cases} w_{k,k} = \int_{x_{k-1}}^{x_k} \frac{1}{(x_k-s)^\alpha} u_{k-1}(s)ds = \frac{h^{1-\alpha}}{(1-\alpha)(2-\alpha)} \ , \\[2em] w_{k,i} = \int_{x_{i-1}}^{x_i} \frac{1}{(x_k-s)^\alpha} u_{i-1}(s)ds - \int_{x_i}^{x_{i+1}} \frac{1}{(x_k-s)^\alpha} u_{i+1}(s)ds \\[2em] \qquad = [(k-i+1)^{2-\alpha} - 2(k-i)^{2-\alpha} + (k-i-1)^{2-\alpha}]w_{k,k}, \quad i = 1,\ldots,k-1 \ , \\[2em] w_{k,0} = -\int_{x_0}^{x_1} \frac{1}{(x_k-s)^\alpha} u_1(s)ds \\[2em] \qquad = \frac{x_k^{1-\alpha}}{1-\alpha} - [k^{2-\alpha} - (k-1)^{2-\alpha}]w_{k,k} \end{cases}$$

Equation (2.6) is a nonsingular triangular system for Y_i', because $w_{k,k} \neq 0$ for $k = 1,2,\ldots$. Note that from (2.7) $w_{k,i} = w_{k-1,i-1}$ for $i = 2,3,\ldots,k$. To compute Y_k' by (2.6) at each step we have only to compute the coefficients $w_{k,1}, w_{k,0}$ and $f'(x_k) - y(0)x_k^{-\alpha}$.

Since

$$y(0) = \lim_{x\to 0} (1-\alpha) \frac{f(x)}{x^{1-\alpha}} \ ,$$

and

$$y'(0) = \lim_{x\to 0} (1-\alpha) \frac{f'(x) - y(0)x^{-\alpha}}{x^{1-\alpha}} \ ,$$

which exist, whenever (1.1) has a solution in $C^1[0,1]$, we take $Y_0 = y(0)$ and $Y_0' = y'(0)$. The values Y_i', Y_i (i = 1,2,\ldots) can then be determined successively by (2.6) together with (2.4). Note that this approach requires no starting procedure.

An estimate of $y''(x)$ is given by the second derivative of (2.1). If we denote this (constant) estimate of $y''(x)$ in $x_i \leq x < x_{i+1}$ by Y_i'', this gives

(2.8) $\qquad Y_i'' = \frac{1}{h} (Y_{i+1}' - Y_i'), \quad x_i \leq x < x_{i+1}$.

3. **Convergence Results.** The proof of Theorem 3.1 requires the following lemma which is a consequence of the standard results for regular infinite system of algebraic equations by Kantorovich and Krylov in [6, p. 27].

Lemma 3.1. If there exist a constant $c > 0$ and an integer $N \geq 1$, all independent of k, such that

$$|x_i| \leq c, \qquad\qquad\qquad i = 0, 1, \ldots, N$$

$$|x_{k+1}| \leq \sum_{i=0}^{k} |\alpha_{k+1,i}| |x_i| + |\beta_k|, \qquad k = N, N+1, \ldots,$$

with

$$\rho_k = 1 - \sum_{i=0}^{k} |\alpha_{k+1,i}| > 0,$$

$$|\beta_k| \leq c \rho_k, \qquad\qquad\qquad k = N, N+1, \ldots,$$

then

$$|x_i| \leq c, \qquad\qquad\qquad i = 0, 1, 2, \ldots.$$

The proof of this lemma can be found in [4, p. 3].

Let $y(x)$ be the exact solution of (1.1) and define the discretization error function $\varepsilon(x)$ by $\varepsilon(x) = y(x) - P(x)$, where $P(x)$ is the quadratic spline approximation to $y(x)$ obtained from our numerical method. Denote $\varepsilon^{(r)}(x_i)$ by $\varepsilon_i^{(r)}$ ($r = 0, 1, 2$). We state the following theorem for the asymptotic estimate of ε_k':

Theorem 3.1. If $y \in C^4[0,1]$ and $2 - \frac{\ln 3}{\ln 2} \leq \alpha < 1$, then for $k = 1, 2, \ldots$

$$(3.1) \qquad \varepsilon_k' = \frac{h^2}{12} y'''(x_k) + O(h^2/k^{1-\alpha}).$$

Proof: Since by assumption $y(x) \in C^4[0,1]$, it is not difficult to show that, for $x \in [x_i, x_{i+1}]$,

$$(3.2) \qquad \varepsilon(x) = \varepsilon_i + \frac{h}{2} [(2u_i(x) - u_i^2(x)) \varepsilon_i' + u_i^2(x) \varepsilon_{i+1}'] + \frac{h^3}{12} y'''(x_i) [2u_i^3(x) - 3u_i^2(x)] + \rho(x),$$

where

$$(3.3) \qquad \rho(x) = \frac{h^3}{12} [2 \int_{x_i}^{x} (\frac{x-s}{h})^3 y^{(4)}(s) ds - 3u_i^2(x) \int_{x_i}^{x_{i+1}} u_{i+1}^2(s) y^{(4)}(s) ds],$$

with $u_i(x)$ as defined in (2.1).

Differentiating (3.2) we obtain the derivative of $\varepsilon(x)$,

$$(3.4) \qquad \varepsilon'(x) = [u_i(x)\varepsilon'_{i+1} - u_{i+1}(x)\varepsilon'_i] + \frac{h^2}{2} y'''(x_i)u_i(x)u_{i+1}(x) + \rho'(x),$$

$$x_i \leq x \leq x_{i+1} \, ,$$

where $\rho'(x)$ can be represented in Lagrange form as

$$(3.5) \qquad \rho'(x) = \frac{h^3}{6} [u_i^3(x)y^{(4)}(\eta(x)) - u_i(x)y^{(4)}(\eta(x_{i+1}))] \quad x_i \leq \eta(x) \leq x_{i+1} \, .$$

Adding and subtracting $-\frac{h^2}{12} [u_i(x)y'''(x_{i+1}) - u_{i+1}(x)y'''(x_i)]$ to the right side of

(3.4) and using the Taylor series expansion for $y'''(x)$ at x_i for $x = x_{i+1}$, it is not

difficult to show that

$$(3.6) \qquad \varepsilon'(x) = [u_i(x)(\varepsilon'_{i+1} - \frac{h^2}{12} y'''(x_{i+1})) - u_{i+1}(x)(\varepsilon'_i - \frac{h^2}{12} y'''(x_i))] + \Phi(x) \, ,$$

$$x_i \leq x \leq x_{i+1} \, ,$$

where

$$(3.7) \qquad \Phi(x) = \frac{h^2}{2} y'''(x_i)[u_i(x)u_{i+1}(x) + \frac{1}{6}] + [\rho'(x) + \frac{h^3}{12} u_i(x)y^{(4)}(\xi(x_{i+1}))] \, ,$$

$$x_i \leq \xi(x) \leq x_{i+1} \, .$$

Since both $y'(x)$ and $P'(x)$ satisfy (2.3) at each knot $x = x_k$,

$$(3.8) \qquad \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{1}{(x_k-s)^\alpha} \varepsilon'(s)ds = 0 \, , \qquad k = 1,2,\ldots \, .$$

Define

$$(3.9) \qquad \hat{\varepsilon}'_i = (i+1)^{1-\alpha}(\varepsilon'_i - \frac{h^2}{12} y'''(x_i)) \, ,$$

we can rewrite (3.8) as

$$(3.10) \qquad \sum_{i=0}^{k} w_{k,i} \frac{1}{(i+1)^{1-\alpha}} \hat{\varepsilon}'_i = \hat{R}_k \, , \qquad k = 1,2,\ldots \, ,$$

where

$$(3.11) \qquad \hat{R}_k = - \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{1}{(x_k-s)^\alpha} \Phi(s)ds \, ,$$

and the $w_{k,i}$'s are defined in (2.7).

Now multiply (3.10) by k^α, difference the resulting equation for k and $k+1$, then

divide by $\frac{(k+1)^\alpha w_{k+1,k+1}}{(k+2)^{1-\alpha}}$ to yield the required error equation:

$$(3.12) \qquad \hat{\varepsilon}'_{k+1} = \sum_{i=0}^{k} \hat{a}_{k+1,i} \hat{\varepsilon}'_i + \hat{b}_k \, , \qquad k = 1,2,\ldots \, ,$$

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where

(3.13)
$$\hat{a}_{k+1,i} = \left(\frac{k+2}{i+1}\right)^{1-\alpha} a_{k+1,i} , \qquad i = 0,1,\ldots,k ,$$

and

(3.14)
$$\hat{b}_k = \frac{(k+2)^{1-\alpha}}{(k+1)^{\alpha} w_{k+1,k+1}} [(k+1)^{\alpha} \hat{R}_{k+1} - k^{\alpha} \hat{R}_k] , \qquad k = 1,2,\ldots ,$$

with

(3.15)
$$a_{k+1,i} = \frac{k^{\alpha} w_{k,i} - (k+1)^{\alpha} w_{k+1,i}}{(k+1)^{\alpha} w_{k+1,k+1}} .$$

Equation (3.12) implies that

(3.16)
$$|\hat{\varepsilon}'_{k+1}| \leq \sum_{i=0}^{k} |\hat{a}_{k+1,i}||\hat{\varepsilon}'_i| + |\hat{b}_k|, \qquad k = 1,2,\ldots .$$

Since $\hat{\varepsilon}'_0 = \varepsilon'_0 - \frac{h^2}{12} y'''(x_0) = O(h^2)$, it is easily shown from Equation (3.10) by using

Lemma 3.3(a) below that $\hat{\varepsilon}'_i = O(h^2)$ for $i = 1,\ldots,\hat{K}$, with $\hat{K}$ as defined in Lemma 3.2

below. On the basis of this together with Lemma 3.2(b) and Lemma 3.3(c) below we can apply

Lemma 3.1 on (3.16) and conclude that

$$\hat{\varepsilon}'_k = (k+1)^{1-\alpha} (\varepsilon'_k - \frac{h^2}{12} y'''(x_k)) = O(h^2), \qquad k = 0,1,2,\ldots .$$

Thus the proof of Theorem 3.1 is completed.

<u>Theorem 3.2.</u> Let the assumptions of Theorem 3.1 be satisfied, then for $k = 1,2,\ldots$

(3.17)
$$|\varepsilon_k| = O(k^{\alpha} h^3) .$$

<u>Proof:</u> Setting $x = x_{i+1}$ in (3.2) and noting that $\rho(x_{i+1}) = O(h^4)$, we obtain

(3.18)
$$\varepsilon_{i+1} = \varepsilon_i + \frac{h}{2}(\varepsilon'_i + \varepsilon'_{i+1}) - \frac{h^3}{12} y'''(x_i) + O(h^4) .$$

Since $\varepsilon_0 = 0$, by means of (3.1) and using the fact that $\sum_{i=1}^{k} \frac{1}{i^{1-\alpha}} \leq \frac{1}{\alpha} k^{\alpha}$ for $0 < \alpha < 1$

and $k \geq 1$, (3.17) follows immediately from (3.18) by induction.

<u>Theorem 3.3.</u> Let the assumptions of Theorem 3.1 be satisfied, then for $k = 0,1,2,\ldots$

(3.19)
$$\varepsilon''_k = - \frac{h}{2} y'''(x_k) + O(h/(k+1)^{1-\alpha}).$$

<u>Proof:</u> Differentiating (3.2) twice and letting $i = k$ we have for $x_k \leq x < x_{k+1}$

(3.20)
$$\varepsilon''(x) = \frac{1}{h}[\varepsilon'_{k+1} - \varepsilon'_k] + y'''(x_k)(x - x_k - \frac{h}{2}) + O(h^2) .$$

Setting $x = x_k$ in (3.20) and using (3.1), (3.19) immediately follows.

In the following we can show that the approximate solution $P(x)$ along with its first and second derivatives converges to the corresponding exact solutions at each point in the interval of integration.

Theorem 3.4. Let the assumptions of Theorem 3.1 be satisfied, then for any fixed $x \in [0,1]$

$$|\varepsilon(x)| = O(h^{3-\alpha}) , \quad |\varepsilon'(x)| = O(h^2), \quad |\varepsilon''(x)| = O(h) .$$

Proof: By means of (3.1) and (3.17), Theorem 3.4 follows immediately from Equations (3.2), (3.4) and (3.20).

The result of Theorem 3.4 has been obtained under the assumptions that $y \in C^4[0,1]$ and that $2 - \frac{\ln 3}{\ln 2} \leq \alpha < 1$. If we relax the assumption on y , viz, y''' is Lipschitz continuous in $[0,1]$, we can show that convergence for the approximate solution and its first two derivatives is still possible, the order of convergence being 2, 2 and 1 , respectively. We can also relax the assumption on α by modifying the structure of the $\hat{a}_{k+1,i}$'s in the error equation (3.12) using a technique analogous to that of [10]. But it will involve a complicated analysis. Since our proof already includes the important case $\alpha = \frac{1}{2}$, an extension of Theorem 3.4 in this direction will not be pursued here. Numerical experiment in Section 4 indicates that the result of Theorem 3.4 holds for all α , $0 < \alpha < 1$.

Lemma 3.2. If the $\hat{a}_{k+1,i}$'s are defined by (3.13) and $2 - \frac{\ln 3}{\ln 2} \leq \alpha < 1$, then there exist integers K and $\hat{K} \geq 1$, independent of h and k , dependent on α , such that

(a) $\qquad \hat{a}_{k+1,i} \geq 0$, $\qquad\qquad i = 0,1,\ldots,k-1, \ k \geq 1$,

$\qquad\qquad \hat{a}_{k+1,k} \geq 0$, $\qquad\qquad k \geq K$,

(b) $\qquad 1 - \sum_{i=0}^{k} |\hat{a}_{k+1,i}| \geq (2-\alpha)(1-\alpha) \frac{(1-c)}{(k+1)^{\alpha}}$, $\quad k \geq \hat{K}$,

for an appropriate c , $\alpha \leq c < 1$.

Proof of (a): From (3.15), using (2.7), we have

$$a_{k+1,0} = \frac{1}{(k+1)^{\alpha} w_{k+1,k+1}} \int_{x_0}^{x_1} \left[\frac{1}{(1-\frac{s}{x_k})^{\alpha}} - \frac{1}{(1-\frac{s}{x_{k+1}})^{\alpha}} \right] \frac{(x_1-s)}{h^{1+\alpha}} \, ds \geq 0$$

since the integrand is nonnegative. Similarly, we can prove that $a_{k+1,i} \geq 0$ for $i = 1, \ldots, k-1$. Finally, it is easy to show that as k increases $a_{k+1,k}$ tends to $3 - 2^{2-\alpha}$, which is greater than or equal to zero for $2 - \frac{\ln 3}{\ln 2} \leq \alpha < 1$, therefore there exists an integer K , independent of h and k such that $a_{k+1,k} \geq 0$ for $k \geq K$ and

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$2 - \frac{\ln 3}{\ln 2} \leq \alpha < 1$. Thus, part (a) follows immediately from (3.13).

Proof of (b): Since it would be complex if we estimate $\sum_{i=0}^{k} \hat{a}_{k+1,i}$ directly, we introduce $\tilde{a}_{k+1,i}$:

$$(3.21) \qquad \tilde{a}_{k+1,i} = \frac{(\frac{k+c}{k})^{1-\alpha} w_{k,i} - w_{k+1,i}}{w_{k+1,k+1}} , \quad 0 \leq c < 1 , \quad i = 0,1,\ldots,k .$$

By noting that $(\frac{k+c}{k})^{1-\alpha} > (\frac{k}{k+1})^{\alpha}$ for $k \geq 1$, it is obvious that for $2 - \frac{\ln 3}{\ln 2} \leq \alpha < 1$,

$$(3.22) \qquad \tilde{a}_{k+1,i} > a_{k+1,i} \geq 0 , \qquad i = 0,1,\ldots,k-1, \ k \geq 1 ,$$

$$\tilde{a}_{k+1,k} > a_{k+1,k} \geq 0 , \qquad k \geq K ,$$

where the K is the same K as defined in part (a) .

Since from (2.7) we have

$$(3.23) \qquad \sum_{i=0}^{k} w_{k,i} = \int_{0}^{x_k} \frac{1}{(x_k - s)^\alpha} ds = \frac{x_k^{1-\alpha}}{1-\alpha}$$

for $k = 1,2,\ldots$, it can easily be shown that for $2 - \frac{\ln 3}{\ln 2} \leq \alpha < 1$,

$$(3.24) \qquad 1 - \sum_{i=0}^{k} |\tilde{a}_{k+1,i}| \geq (2-\alpha)(1-\alpha) \frac{(1-c)}{(k+1)^\alpha} , \qquad k \geq K .$$

If we can show that for an appropriate c , $0 \leq c < 1$, $\tilde{a}_{k+1,i} - \hat{a}_{k+1,i} \geq 0$, then the proof is completed. To do this, it is sufficient to show that, for $i = 0,1,\ldots,k$

$$D_{k+1,i} = [(\frac{k+c}{k})^{1-\alpha}(\frac{k+1}{k})^\alpha - 1][k^\alpha w_{k,i}] - [(\frac{k+2}{i+1})^{1-\alpha} - 1][k^\alpha w_{k,i} - (k+1)^\alpha w_{k+1,i}] \geq 0 .$$

By using (2.7) and letting $s = ht$, it is easily verified that for $i = 1,\ldots,k-3$,

$$(3.25) \qquad D_{k+1,i} \geq [\int_{i-1}^{i} L_{k,i}(t)(t-i+1)dt + \int_{i}^{i+1} L_{k,i}(t)(i+1-t)dt]h^{1-\alpha} ,$$

where

$$L_{k,i}(t) = [(\frac{k+c}{k})^{1-\alpha}(\frac{k+1}{k})^\alpha - 1] \frac{k^\alpha}{(k-t)^\alpha} - [(\frac{k+2}{i+1})^{1-\alpha} - 1] \frac{\alpha t}{k^{1-\alpha}(k-t)^{1-\alpha}} .$$

Since for $i-1 \leq t \leq i+1$, $i = 1,\ldots,k-3$ and $\alpha \leq c < 1$,

$$L_{k,i}(t) \geq \frac{2\alpha^2}{k^2} > 0$$

when k is sufficiently large, it is obvious from (3.25) that $D_{k+1,i} \geq 0$ for $i = 1, \ldots, k-3$. Also when $\alpha \leq c < 1$ and $2 - \frac{\ln 3}{\ln 2} \leq \alpha < 1$, it is easy to show by using

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(2.7) that $D_{k+1,i} \geq 0$ $(i = 0,k-2,k-1,k)$ for sufficiently large k .

Thus for $i = 0,1,\ldots,k$, there exists an integer $\tilde{K} \geq 1$, independent of h and k, dependent on α , such that

$$\tilde{a}_{k+1,i} \geq \hat{a}_{k+1,i} \ , \qquad k \geq \tilde{K} \ .$$

By means of part (a), (3.22), (3.24), it follows that

$$1 - \sum_{i=0}^{k} |\hat{a}_{k+1,i}| \geq 1 - \sum_{i=0}^{k} |\tilde{a}_{k+1,i}| \geq (2-\alpha)(1-\alpha) \frac{1-c}{(k+1)^{\alpha}}$$

for $\alpha \leq c < 1$, $2 - \frac{\ln 3}{\ln 2} \leq \alpha < 1$, and $k \geq \hat{K}$ with $\hat{K} = \max(K,\tilde{K})$.

<u>Lemma 3.3.</u> If $y \in C^4[0,1]$, then there exist constants $\hat{c}_1, \hat{c}_2, \hat{c}_3 > 0$, independent of h and k , dependent on α , such that

(a) $|\hat{R}_k| \leq \hat{c}_1 h^{3-\alpha}$,

(b) $|\hat{R}_{k+1} - \hat{R}_k| \leq \hat{c}_2 \dfrac{h^{3-\alpha}}{k}$,

(c) $|\hat{b}_k| \leq \hat{c}_3 \dfrac{h^2}{k^{\alpha}}$,

for $k = 1,2,\ldots$, where $\hat{R}_k$ and $\hat{b}_k$ are defined in (3.11) and (3.14), respectively.

<u>Proof of (a)</u>: By repeated integration by parts, it is not difficult to show that for $i = 0,1,\ldots,k-1$

(3.26)
$$\frac{h^2}{2} \int_{x_i}^{x_{i+1}} \frac{1}{(x_k-s)^{\alpha}} y'''(x_i) [u_i(s)u_{i+1}(s) + \frac{1}{6}] ds$$

$$= \frac{\alpha(1+\alpha)}{24} h^4 \int_{x_i}^{x_{i+1}} \frac{1}{(x_k-s)^{2+\alpha}} y'''(x_i) u_i^2(s) u_{i+1}^2(s) ds \ .$$

Using (3.26) we can rewrite (3.11) as

(3.27) $\hat{R}_k = \hat{A}_k^{(1)} + \hat{A}_k^{(2)}$, $k = 1,2,\ldots$,

where

$$\hat{A}_k^{(1)} = - \frac{\alpha(1+\alpha)}{24} h^4 \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{1}{(x_k-s)^{2+\alpha}} y'''(x_i) u_i^2(s) u_{i+1}^2(s) ds \ .$$

$$\hat{A}_k^{(2)} = - \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} \frac{1}{(x_k-s)^{\alpha}} [\rho'(s) + \frac{h^3}{12} u_i(s) y^{(4)}(\xi(x_{i+1}))] ds \ ,$$

with $\rho'(s)$ as defined in (3.5).

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Let $M_3 = \max_{x \in [0,1]} |y'''(x)|$ and $M_4 = \max_{x \in [0,1]} |y^{(4)}(x)|$. Then by straightforward

estimation and noting that $hk \leq 1$, we obtain from (3.27)

$$(3.28) \qquad |\hat{R}_k| \leq [\frac{\alpha(1+\alpha)}{24} (\frac{1}{1-\alpha} + \frac{1}{16} \frac{1}{1+\alpha})] M_3 h^{3-\alpha} + [\frac{5}{12(1-\alpha)}] M_4 h^{4-\alpha} k^{1-\alpha}$$

$$\leq \hat{c}_1 h^{3-\alpha},$$

where $\hat{c}_1 = [\frac{\alpha(1+\alpha)}{24} (\frac{1}{1-\alpha} + \frac{1}{16} \frac{1}{1+\alpha})] M_3 + \frac{5}{12(1-\alpha)} M_4$.

<u>Proof of (b)</u>: Subtraction of (3.27) from (3.27) with k replaced by $k+1$, then by

straightforward estimation and noting that $hk \leq 1$, it is not difficult to show that

for $k = 1,2,\ldots$

$$(3.29) \qquad |\hat{R}_{k+1} - \hat{R}_k| \leq |\hat{A}_{k+1}^{(1)} - \hat{A}_k^{(1)}| + |\hat{A}_{k+1}^{(2)} - \hat{A}_k^{(2)}|$$

$$\leq \{\frac{\alpha(1+\alpha)}{24}[\frac{1}{1-\alpha} + \frac{1}{16} \frac{1}{1+\alpha}] M_4 h^{4-\alpha} + \frac{\alpha(1+\alpha)}{24} \frac{1}{16} M_3 \frac{h^{3-\alpha}}{k^{2+\alpha}} \}$$

$$+ \{\frac{5}{12} (1 + \frac{3}{1-\alpha}) M_4 h^{4-\alpha} \}$$

$$\leq \hat{c}_2 \frac{h^{3-\alpha}}{k},$$

where $\hat{c}_2 = \frac{\alpha(1+\alpha)}{384} M_3 + \{\frac{\alpha(1+\alpha)}{24} [\frac{1}{1-\alpha} + \frac{1}{16} \frac{1}{1+\alpha}] + \frac{5}{12} (1 + \frac{3}{1-\alpha}) \} M_4$.

<u>Proof of (c)</u>: From (3.14), using (3.28), (3.29) and (2.7), we obtain for $k = 1,2,\ldots$

$$|\hat{b}_k| \leq \frac{(k+2)^{1-\alpha}}{w_{k+1,k+1}} \{|\hat{R}_{k+1} - \hat{R}_k| + [1 - (\frac{k}{k+1})^{\alpha}] |\hat{R}_k|\}$$

$$\leq \hat{c}_3 \frac{h^2}{k^{\alpha}},$$

where $\hat{c}_3 = 3(1-\alpha)(2-\alpha)(\alpha \hat{c}_1 + \hat{c}_2)$.

4. **Numerical Examples.** The quadratic spline method described in Section 2 was applied to the following problems:

Problem 4.1.

$$\int_0^x \frac{1}{\sqrt{x-s}}\, y(s)\,ds = \frac{2}{\sqrt{1+x}}\, \log\left(\sqrt{1+x} + \sqrt{x}\right), \qquad 0 \le x \le 1 .$$

The solution is $y(x) = \frac{1}{1+x}$.

Problem 4.2.

$$\int_0^x \frac{1}{(x-s)^\alpha}\, y(s)\,ds = f(x), \qquad\qquad 0 \le x \le 3,$$

where

$$f(x) = \frac{6}{(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)}\, x^{4-\alpha} + \frac{2}{(1-\alpha)(2-\alpha)}\, x^{2-\alpha} - \frac{4}{1-\alpha}\, x^{1-\alpha},$$

with $\alpha = 0.1, 0.5, 0.9$. The solution is $y(x) = x^3 + 2x - 4$.

In Table 4.1(a) and Table 4.1(b) we list the error $\epsilon(x)$, $\epsilon'(x)$, and $\epsilon''(x)$ of Problem 4.1 at knots and at mid-points between the knots, respectively on $[0,1]$ for different stepsizes h. The error $\epsilon(x)$, $\epsilon'(x)$ and $\epsilon''(x)$ satisfy the predicted $h^{5/2}$, h^2 and h dependence, and the error $\epsilon''(x)$ at the mid-point is actually smaller than that at the knot, which is obvious from Equation (3.20). Asymptotic estimates of $\epsilon'(x)$ and $\epsilon''(x)$ at knots computed from Equations (3.1) and (3.19), respectively, are enclosed in brackets in Table 4.1(a).

In Table 4.2 we list the errors $\epsilon(x)$, $\epsilon'(x)$ and $\epsilon''(x)$ of Problem 4.2 at knots on $[0,3]$ for different stepsizes h. The errors obtained for $\alpha = 0.1$, for which convergence is not guaranteed by Theorem 3.4, satisfy the same order of convergence as the errors obtained for $\alpha = 0.5$ and $\alpha = 0.9$.

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Table 4.1(a)

Error at knots for Problem 4.1 (h=0.1)

x	$\epsilon(x)$			$\epsilon'(x)$			$\epsilon''(x)$		
	h	h/3	h/9	h	h/3	h/9	h	h/3	h/9
0.2	.1142E-3	.7572E-5	.4759E-6	-.1691E-2	-.2653E-3	-.2959E-4 (-.2977E-4)	.1294E 0	.4726E-1	.1597E-1 (.1608E-1)
0.4	.1149E-3	.7244E-5	.4609E-6	-.1298E-2	-.1478E-3	-.1626E-4 (-.1607E-4)	.7362E-1	.2561E-1	.8631E-2 (.8677E-2)
0.6	.1033E-3	.6520E-5	.4161E-6	-.8118E-3	-.8833E-4	-.9646E-5 (-.9419E-5)	.4383E-1	.1505E-1	.5063E-2 (.5086E-2)
0.8	.9230E-4	.5840E-5	.3730E-6	-.5232E-3	-.5602E-4	-.6079E-5 (-.5880E-5)	.2756E-1	.9411E-2	.3162E-2 (.3175E-2)
1.0	.8319E-4	.5268E-5	.3365E-6	-.3519E-3	-.3729E-4	-.4023E-5 (-.3858E-5)	.1816E-1	.6183E-2	.2076E-2 (.2083E-2)

Table 4.1(b)

Error at Mid-points for Problem 4.1 (h=0.1)

x	$\epsilon(x)$			$\epsilon'(x)$			$\epsilon''(x)$		
	h	h/3	h/9	h	h/3	h/9	h	h/3	h/9
0.25	.1235E-3	.7520E-5	.4788E-6	.1153E-2	.1131E-3	.1264E-4	.4000E-2	.3366E-3	.3783E-4
0.45	.1106E-3	.7033E-5	.4499E-6	.5196E-3	.5926E-4	.6762E-5	.7868E-3	.1698E-3	.1899E-4
0.65	.9901E-4	.6322E-5	.4046E-6	.2813E-3	.3395E-4	.3941E-5	.7758E-3	.9293E-4	.1022E-4
0.85	.8900E-4	.5676E-5	.3631E-6	.1658E-3	.2072E-4	.2444E-5	.4970E-3	.5402E-4	.5871E-5

Table 4.2

Error at Knots for Problem 4.2 (h=0.1)

α	x	ε(x)			ε'(x)			ε"(x)		
		h	h/3	h/9	h	h/3	h/9	h	h/3	h/9
0.1	0.0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	-.2069E 0	-.6897E-1	-.2299E-1
	0.5	-.2405E-4	-.5397E-6	-.2366E-7	.7111E-2	.5952E-3	.6174E-4	-.3388E 0	-.1022E 0	-.3334E-1
	1.0	-.1139E-4	-.6115E-6	-.2525E-7	.4129E-2	.5527E-3	.6173E-4	-.2840E 0	-.9985E-1	-.3333E-1
	1.5	-.1727E-4	-.6389E-6	-.2626E-7	.5357E-2	.5557E-3	.6173E-4	-.3066E 0	-.1000E 0	-.3333E-1
	2.0	-.1540E-4	-.6559E-6	-.2701E-7	.4852E-2	.5555E-3	.6173E-4	-.2973E 0	-.1000E 0	-.3333E-1
	2.5	-.1663E-4	-.6700E-6	-.2761E-7	.5060E-2	.5555E-3	.6173E-4	-.3011E 0	-.1000E 0	-.3333E-1
	3.0	-.1651E-4	-.6818E-6	-.2811E-7	.4975E-2	.5555E-3	.6173E-4	-.2995E 0	-.1000E 0	-.3333E-1
0.5	0.0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	-.2400E 0	-.8000E-1	-.2667E-1
	0.5	-.3047E-3	-.1891E-4	-.1201E-5	.4743E-2	.5372E-3	.6054E-4	-.3000E 0	-.9998E-1	-.3333E-1
	1.0	-.4202E-3	-.2654E-4	-.1693E-5	.4799E-2	.5425E-3	.6089E-4	-.2999E 0	-.9999E-1	-.3333E-1
	1.5	-.5106E-3	-.3259E-4	-.2072E-5	.4835E-2	.5449E-3	.6104E-4	-.2999E 0	-.1000E 0	-.3333E-1
	2.0	-.5873E-3	-.3738E-4	-.2392E-5	.4857E-2	.5463E-3	.6113E-4	-.3000E 0	-.1000E 0	-.3333E-1
	2.5	-.6551E-3	-.4176E-4	-.2673E-5	.4871E-2	.5473E-3	.6119E-4	-.3000E 0	-.1000E 0	-.3333E-1
	3.0	-.7165E-3	-.4585E-4	-.2928E-5	.4882E-2	.5479E-3	.6124E-4	-.3000E 0	-.1000E 0	-.3333E-1
0.9	0.0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	-.2857E 0	-.9524E-1	-.3175E-1
	0.5	-.1744E-2	-.1723E-3	-.1717E-4	.1899E-2	.2466E-3	.3096E-4	-.2994E 0	-.9994E-1	-.3332E-1
	1.0	-.3236E-2	-.3210E-3	-.3192E-4	.2105E-2	.2672E-3	.3302E-4	-.2997E 0	-.9997E-1	-.3333E-1
	1.5	-.4653E-2	-.4621E-3	-.4597E-4	.2219E-2	.2786E-3	.3416E-4	-.2998E 0	-.9998E-1	-.3333E-1
	2.0	-.6023E-2	-.5985E-3	-.5955E-4	.2298E-2	.2865E-3	.3494E-4	-.2999E 0	-.9998E-1	-.3333E-1
	2.5	-.7359E-2	-.7315E-3	-.7279E-4	.2357E-2	.2924E-3	.3553E-3	-.2999E 0	-.9999E-1	-.3333E-1
	3.0	-.8668E-2	-.8618E-3	-.8577E-4	.2405E-2	.2972E-3	.3600E-4	-.2999E 0	-.9999E-1	-.3333E-1

5. <u>Divergence of Higher Order Spline Methods</u>.   We have shown that the linear spline method (see Appendix) and the quadratic spline method (see Section 2) based on using the differentiated form of Equation (1.1) alone are convergent.   It might be supposed that the same technique applied to cubic and higher order splines would result in higher order approximation methods for Equation (1.1).   Such an approach, however, fails, because of instability, in the case that the spline is of full continuity.   For illustration, apply the method by using a cubic spline in $C^2[0,1]$ to the following Abel integral equation:

$$\int_0^x \frac{1}{(x-s)^\alpha} \, y(s) \, ds = \frac{6}{(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)} \, x^{4-\alpha},$$

where $y(x) = x^3$, with $\alpha = 0.1$, 0.5 and 0.9.   The divergence of the numerical results can be seen very clearly in Table 5.1.

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Table 5.1

Divergence of Numerical Results

By the Cubic Spline Method (h=0.1/3)

α	x	$\epsilon(x)$		$\epsilon'(x)$		$\epsilon''(x)$		$\epsilon'''(x)$	
		h	h/3	h	h/3	h	h/3	h	h/3
0.1	0.0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	-.2051E 0	-.6838E-1
	0.2	.1986E-4	.4796E 0	-.1525E-1	-.1106E 4	-.1693E 1	-.3688E 6	.2206E 3	.1442E 9
	0.4	.2776E-1	.1314E10	-.2134E 2	-.3029E13	-.2372E 4	-.1011E16	.3092E 6	.3951E18
	0.6	.3884E 2	.3600E19	-.2986E 5	-.8300E22	-.3320E 7	-.2769E25	.4326E 9	.1082E28
	0.8	.5435E 5	.9862E28	-.4178E 8	-.2274E32	-.4645E10	-.7585E34	.6054E12	.2966E37
	1.0	.7605E 8	.2702E38	-.5846E11	-.6230E41	-.6499E13	-.2078E44	.8471E15	.8125E46
0.5	0.0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	-.2286E 0	-.7619E-1
	0.2	.8032E-5	.6305E-3	-.7850E-3	-.1887E 0	-.1324E 0	-.9696E 2	-.1203E 2	.2686E 5
	0.4	.6344E-3	.3292E 3	-.6326E-1	-.9853E 5	-.1083E 2	-.5064E 8	.1000E 4	.1403E11
	0.6	.5107E-1	.1719E 9	-.5094E 1	-.5145E11	-.8727E 3	-.2644E14	.8058E 5	.7326E16
	0.8	.4112E 1	.8979E14	-.4102E 3	-.2687E17	-.7027E 5	-.1381E20	.6489E 7	.3826E22
	1.0	.3312E 3	.4689E20	-.3304E 5	-.1403E23	-.5659E 7	-.7212E25	.5226E 9	.1998E28
0.9	0.0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	.0000E 0	-.2581E 0	-.8602E-1
	0.2	.5629E-6	.5083E-7	-.6916E-5	-.2057E-5	-.3507E-2	-.4332E-2	.2554E-1	.7645E 0
	0.4	.1610E-5	.9007E-6	-.2017E-4	-.4023E-4	-.1314E-1	-.8809E-1	.6043E 0	.1719E 2
	0.6	.4117E-5	.1719E-4	-.5555E-4	-.7766E-3	-.3899E-1	-.1704E 1	.2294E 1	.3340E 3
	0.8	.1056E-4	.3314E-3	-.1503E-3	-.1498E-1	-.1083E 0	-.3288E 2	.6824E 1	.6447E 4
	1.0	.2758E-4	.6394E-2	-.4045E-3	-.2891E 0	-.2942E 0	-.6343E 3	.1898E 2	.1244E 6

Solving a Class of Abel Integral Equations
By Linear Splines

Introduction.   In this appendix we consider the use of a linear spline to obtain a
global approximate solution for Equation (1.1). We shall describe the method briefly
and state the important theorems. The proofs of the theorems have been carried through
but the details are omitted, since the methods of proof are similar to those already used
in this paper.

A Linear Spline Method.   Let $x_i = ih$, $i = 0,1,\ldots,$ where h is an arbitrary constant
stepsize. Let Y_i, Y_i' denote approximations to $y(x_i)$, $y'(x_i)$, respectively. We use a
linear spline function $P(x)$ in $C[0,1]$ with knots at the points x_i as an approxima-
tion to $y(x)$, i.e., for $i = 0,1,2,\ldots$.

$$(A.1) \qquad P(x) = Y_i + Y_i'(x-x_i), \qquad x_i \le x \le x_{i+1} .$$

The function $P(x)$ is continuous at the knots.

The approximate solution $P(x)$ of the integral Equation (1.1) is derived from values
$P(x_i) = Y_i$ and $P'(x_i) = Y_i'$. We know how to find Y_k' , assuming that we know Y_i' ,
$i = 0,1,\ldots,k-1$. We can then deduce Y_{k+1} from the equation obtained by setting $x = x_{k+1}$
in (A.1), which gives

$$(A.2) \qquad Y_{k+1} = Y_k + h\, Y_k' .$$

To compute Y_i' , we require that $P'(x)$ satisfy (2.3), i.e., $y'(x)$ is replaced by $P'(x)$
derived from the values $P'(x_i) = Y_i'$ computed from

$$(A.3) \qquad \int_0^{x_{k+1}} \frac{1}{(x_{k+1}-s)^\alpha} P'(s)\,ds = f'(x_{k+1}) - y(0)x_{k+1}^{-\alpha} , \qquad k = 0,1,2,\ldots .$$

This can be rewritten in the form:

$$(A.4) \qquad \sum_{i=0}^{k} \left(\int_{x_i}^{x_{i+1}} \frac{1}{(x_{k+1}-s)^\alpha}\,ds \right) Y_i' = f'(x_{k+1}) - y(0)x_{k+1}^{-\alpha} , \qquad k = 0,1,2,\ldots .$$

Equation (A.4) is a nonsingular triangular system for Y_i' .

For the starting value Y_0, we take

$$(A.5) \qquad Y_0 = y(0) = \lim_{x \to 0} (1-\alpha) \frac{f(x)}{x^{1-\alpha}} ,$$

which exists whenever (1.1) has a continuous solution. The values Y_k', $Y_{k+1}(k = 0,1,...)$ can then be determined successively by (A.4) together with (A.2).

Define the discretization error function $\varepsilon(x)$ by $\varepsilon(x) = y(x) - P(x)$. We state the following theorems:

Theorem A.1. If $y \in C^3[0,1]$, then for $k = 0,1,2,...$

$$(A.6) \qquad \varepsilon'(x_k) = -\frac{h}{2} y''(x_k) + O(h/(k+1)^{1-\alpha})$$

and

$$(A.7) \qquad \varepsilon'(x_k + \frac{h}{2}) = O(h/(k+1)^{1-\alpha}) .$$

Theorem A.2. If the assumption of Theorem A.1 is satisfied, then for $k = 0,1,2,...$

$$(A.8) \qquad \varepsilon(x_k) = O(k^\alpha h^2) .$$

Theorem A.3. If the assumption of Theorem A.1 is satisfied, then for any fixed $x \in [0,1]$

$$(A.9) \qquad |\varepsilon(x)| = O(h^{2-\alpha}), \qquad |\varepsilon'(x)| = O(h) .$$

A Numerical Example. The linear spline method was applied to Problem 4.1. The errors $\varepsilon(x)$, $\varepsilon'(x)$ at knots and at mid-points between the knots are tabulated in Tables A.1(a) and A.1(b), respectively on [0,1] for different stepsizes h. The errors $\varepsilon(x)$ and $\varepsilon'(x)$ satisfy the predicted $h^{3/2}$ and h dependence, respectively, and the error $\varepsilon'(x)$ at mid-point is actually smaller than that at knot, which agrees with the results of Theorem A.1. Asymptotic estimates of $\varepsilon'(x)$ at knots computed from Equation (A.6) are enclosed in brackets in Table A.1(a) .

Table A.1(a)

Error at Knots for Problem 4.1 (h=0.1)

x	$\epsilon(x)$			$\epsilon'(x)$		
	h	h/3	h/9	h	h/3	h/9
0.2	-.3989E-2	-.8823E-3	-.1809E-3	-.5878E-1	-.1984E-1	-.6568E-2 (-.6430E-2)
0.4	-.4750E-2	-.9874E-3	-.1968E-3	-.3462E-1	-.1191E-1	-.4017E-2 (-.4049E-2)
0.6	-.4782E-2	-.9699E-3	-.1912E-3	-.2220E-1	-.7761E-2	-.2646E-2 (-.2713E-2)
0.8	-.4600E-2	-.9209E-3	-.1804E-3	-.1507E-1	-.5339E-2	-.1836E-2 (-.1905E-2)
1.0	-.4360E-2	-.8659E-3	-.1691E-3	-.1067E-1	-.3827E-2	-.1326E-2 (-.1389E-2)

Table A.1(b)

Error at Mid-Points for Problem 4.1 (h=0.1)

x	$\epsilon(x)$			$\epsilon'(x)$		
	h	h/3	h/9	h	h/3	h/9
0.25	-.5529E-2	-.1073E-2	-.2048E-3	-.4332E-2	-.6625E-3	-.1111E-3
0.45	-.5601E-2	-.1080E-2	-.2065E-3	.4357E-4	.8546E-4	.2427E-4
0.65	-.5300E-2	-.1021E-2	-.1956E-3	.1114E-2	.2669E-3	.5568E-4
0.85	-.4937E-2	-.9511E-3	-.1825E-3	.1388E-2	.2987E-3	.6007E-4

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Global approximation methods for solving the Abel integral equation
$$\int_0^x \frac{1}{(x-s)^\alpha} y(s)ds = f(x) \ , \quad 0 < \alpha < 1, \ x \geq 0 \ ,$$
by means of splines with full continuity are considered. The methods are based on using the differentiated form of the above equation. It is shown that the use of linear splines in C leads to a $2-\alpha$ method for $0 < \alpha < 1$ and the use of quadratic splines in C^1 leads to a $3-\alpha$ method, which computational experiments indicate is stable for $0 < \alpha < 1$, though this is proved only for

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$0.415(\doteq 2 - \frac{\ell n \ 3}{\ell n \ 2}) \leq \alpha < 1.$ The same technique applied to cubic and higher-order splines gives rise to divergent methods.